

# VOLTERRA'S INTEGRAL EQUATION OF THE SECOND KIND, WITH DISCONTINUOUS KERNEL\*

BY

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The integral equation of the second kind, of VOLTERRA, is written

$$(1) \quad u(x) = \phi(x) + \int_a^x K(x, \xi) u(\xi) d\xi.$$

In this equation the function  $K(x, \xi)$  is called the kernel; the desired function is  $u(x)$ . When the functions  $K(x, \xi)$  and  $\phi(x)$  are continuous there is no difficulty in finding a continuous  $u(x)$  that shall satisfy equation (1). This general case of the equation (1), under the conditions solely that  $\phi(x)$  be continuous when  $a \leq x \leq b$  and that  $K(x, \xi)$  be continuous in the triangular region  $a \leq \xi \leq x \leq b$ , was first investigated by VOLTERRA,† who showed that there is one and only one continuous solution in the interval  $a \leq x \leq b$ . His method applies without essential change to equations whose kernels are finite in the region  $a \leq \xi \leq x \leq b$  and have discontinuities, provided the discontinuities are regularly‡ distributed.

Let us consider, however, a certain equation to which we are led by a hydrostatical problem. Suppose we are given a tube lying in a vertical plane along a curve of arbitrary shape,  $s = u(x)$ , where  $s$  is the distance along the curve and  $x$  the altitude. Let us fill this tube with a liquid of variable linear density  $\nu$ , and then regulate its height  $x$  in the tube by allowing various amounts to flow through the bottom. Let us then regard  $\nu$  as an analytic function of the depth in the liquid, i. e.  $\nu = \nu(x - \xi)$ . The average linear density is given by the formula

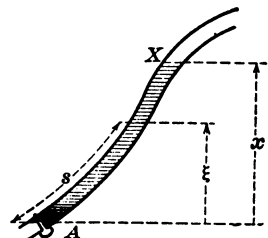


FIG. 1.

$$h(x) = \frac{\int_A^x \nu(x - \xi) ds}{\int_A^x ds} = \frac{\int_0^x \nu(x - \xi) u'(\xi) d\xi}{\int_0^x u'(\xi) d\xi},$$

\* Presented to the Society, September 13, 1909.

† V. VOLTERRA, *Atti dell' Accademia dei Lincei*, ser. 5, vol. 5 (1896), pp. 177-184.

‡ M. BÔCHER: *An Introduction to the Study of Integral Equations*; Cambridge Tracts in Mathematics and Mathematical Physics, No. 10, 1909.

so that, when the curve is given, the average density is determined explicitly. Let us propose to ourselves the inverse problem: given the average linear density, to find the shape of the curve.

In the first place, if  $u'(\xi)$  is continuous  $h(x)$  is continuous, and if we let

$$\begin{aligned} v(x - \xi) &= \alpha + \alpha_1(x - \xi) + \alpha_2(x - \xi)^2 + \dots \\ &= \alpha + G(x - \xi), \end{aligned}$$

where  $G(0) = 0$ , then  $h(x) = \alpha + g(x)$ , where  $g(x)$  is some function that vanishes at least to the first order when  $x = 0$ . We shall look only for solutions  $u(x)$  that have continuous derivatives. Hence we may write the equation to determine  $u'(x)$  as

$$g(x) \int_0^x u'(\xi) d\xi = \int_0^x G(x - \xi) u'(\xi) d\xi,$$

or

$$0 = \int_0^x [G(x - \xi) - g(x)] v(\xi) d\xi,$$

where  $v(\xi) = u'(\xi)$ .

This is an equation of which the kernel vanishes when  $x = \xi = 0$ . If we differentiate once and divide by  $g(x)$ , we get the equation

$$v(x) = \int_0^x \frac{G(x - \xi) - g'(x)}{g(x)} v(\xi) d\xi,$$

which is a homogeneous equation of the form (1), with *discontinuous* kernel.\*

The corresponding extension in the general equation (1) of the second kind is found by differentiation of the general integral equation of the first kind

$$(2) \quad \phi(x) - \phi(a) = \int_a^x K(x, \xi) u(\xi) d\xi.$$

It was originally an equation of this kind,

$$\psi(a) = \int_{x=0}^{x=a} \frac{ds}{(a-x)^n}, \quad 0 < n < 1,$$

by which ABEL in 1823 † introduced the subject of integral equations; an equation of the first kind with discontinuous kernel. The general equation (2) where the kernel is continuous is itself not easy to treat. Yet in his first paper VOLTERRA shows that if  $\phi(y)$  and  $\phi'(y)$  remain continuous when  $a \leq y \leq b$ , and if  $K(x, y)$  and  $\partial K / \partial x = K_1(x, y)$  remain continuous in the triangular

\* This equation will be treated in a later paper.

† VOLTERRA: *Annali di Matematica*, ser. 2, vol. 25 (1897), p. 140.

region  $a \leq y \leq x \leq b$ , while  $K(x, x) \neq 0$ ,  $a \leq x \leq b$ , then there is one and only one solution of (2) continuous in the interval  $a \leq x \leq b$ .\*

There are two manifest ways of enlarging the kernel of (2) beyond VOLTERRA's first restrictions. One way is to generalize the equation of ABEL to the form

$$\phi(x) - \phi(a) = \int_a^x \frac{G(x, \xi)}{(x - \xi)^\lambda} u(\xi) d\xi, \quad \lambda < 1,$$

where  $G(x, \xi)$  and  $\partial G / \partial x$  are continuous in the triangular region and  $G(x, x) \neq 0$ ; this generalization VOLTERRA carries out in the second of his papers † appearing in the *Atti di Torino*, and shows that there is one and only one continuous solution. A second way is to allow the kernel of (2) to vanish when  $x = \xi = a$ . This generalization, with the restriction that  $K(x, \xi)$  shall not vanish to higher order when  $\xi$  and  $x$  are equal and both approach  $a$ , than when  $\xi$  and  $x$  approach  $a$  independently, is considered by VOLTERRA in his third and fourth Turin papers. ‡ And here it turns out that there is in general more than one solution continuous in the neighborhood of  $a$ .

The same problem is considered by E. HOLMGREN § in 1900, and the special case is treated in which the kernel is in the form

$$K(x, y) = a_0 y + a_1 x + K'(x, y)$$

and  $a_0 + a_1$  is not necessarily unequal to zero. The present state of the question is best set forth in a recent paper of LALESCO. || He shows among other things that the equation

$$(3) \quad \int_0^x f(x, s) u(s) ds = F(x),$$

in which

$$f(x, s) = A_n x^n + A_{n-1} x^{n-1} s + \dots + A_0 s^n + \psi(x, s),$$

— here  $\psi(x, s)$  contains only terms whose total degree in  $x$  and  $s$  is greater than  $n$ , and  $A_0 + A_1 + \dots + A_n \neq 0$ , — has a solution  $u(x)$ , finite at  $x = 0$  and depending linearly on  $k$  parameters, provided that  $F(x)$  vanishes to a sufficiently high order and that  $k$  distinct solutions of the equation

\* *Atti della Reale Accademia delle Scienze di Torino*, vol. 31, Jan., 1896, pp. 311-323.

† *Atti della Reale Accademia delle Scienze di Torino*, vol. 31, Jan., 1896, pp. 400-408.

‡ *Atti della Reale Accademia delle Scienze di Torino*, vol. 31, March, 1896, pp. 557-567 and April, 1896, pp. 693-708.

§ E. HOLMGREN: Sur une théorème de M. Volterra sur l'inversion des intégrales définies; *Atti della Reale Acc. delle Scienze di Torino*, vol. 35 (1900), p. 570.

|| T. LALESCO: Sur l'équation de Volterra; *Journal de Mathématiques*, ser. 6, vol. 4 (1908) p. 125.

$$\frac{A_0}{r-1} + \frac{A_1}{r-2} + \cdots + \frac{A_n}{r-n-1} = 0^*$$

have their real parts negative. LALESCO considers also the special case in which

$$f(x, s) = A_1 x + A_0 s + \alpha x^2 + \beta xy + \gamma y^2 + \psi_3(x, s),$$

where  $A_1 + A_0 = 0$  and  $\alpha + \beta + \gamma \neq 0$ , and shows, as was previously announced by E. HOLMGREN (see page 395, note), that if

$$\frac{A_0}{\alpha + \beta + \gamma} > 0$$

there will be a single real solution, while if

$$\frac{A_0}{\alpha + \beta + \gamma} < 0$$

there will be an infinity of solutions depending on a single parameter.

LALESCO's method of procedure, to which the method of our later paper is analogous, is to differentiate the equation (3) in the general case  $n + 1$  times, and then to base on the resulting Fuchsian equations an approximation method by which the given equation may be solved. It is essential in order to reach a Fuchsian equation that  $A_0 + A_1 + \cdots + A_n \neq 0$ .

The equation to which (3) is reduced by differentiating more than once is a special case of a type of equations proposed by BURGATTI,† and investigated in a short note by FUBINI,‡ i. e., equations of the form

$$(4) \quad \int_0^x \left[ f_0(x, s)u(s) + f_1(x, s) \frac{du}{ds} + \cdots + f_n(x, s) \frac{d^n u}{ds^n} \right] ds = F(x).$$

The connection between equations of the first kind with kernel vanishing at the initial point of the interval, and equations of the second kind with kernels discontinuous at the initial point of the interval is seen, as we have said, by differentiating once. Equation (3) thus becomes

$$f(x, x)u(x) + \int_0^x \frac{\partial f(x, s)}{\partial x} u(s) ds = F'(x)$$

or

$$u(x) = \frac{F'(x)}{f(x, x)} - \int_0^x \frac{1}{f(x, x)} \frac{\partial f(x, s)}{\partial x} u(s) ds,$$

\* This equation was originally given by VOLTERRA, and the fact shown that there is but one finite solution if the real parts of all the roots are positive, while otherwise the solution is not determined uniquely.

† BURGATTI: *Rendiconti dell' Accademia dei Lincei*, 2. semestre 1903, p. 443 and p. 596.

‡ FUBINI: *Rendiconti dell' Accademia delle Scienze Fisiche e Matematiche di Napoli*, ser. 3, vol. 10 (1904), pp. 61-64.

which on account of the restriction  $A_0 + A_1 + \dots + A_n \neq 0$  is an equation the kernel of which becomes infinite at  $x = \xi = 0$  to not higher than the first order. The special equation on page 396 corresponds to a case where the kernel of an equation of the second kind becomes infinite at  $x = \xi = 0$  to the second order.

In the present paper we consider equations whose kernels are absolutely integrable, and equations related to them by change of dependent variable. Later we shall treat equations whose kernels, though not of so general a type, are not restricted in their order of becoming infinite, and endeavor to define as to continuity the character of functions that can possibly satisfy the equations under investigation.

Throughout the course of the work it will be convenient to be able to refer to the following two conditions, (A) and (B).

*Condition (A).* A real function of the two variables  $x, \xi$  is to be continuous in the triangle  $T: a \leq \xi \leq x \leq b, b > a \geq 0$  except on a finite number of curves, each composed of a finite number of continuous pieces with continuously turning tangents. Any vertical portion is to be considered a separate piece, and of such pieces there are to be merely a finite number,  $x = \beta_1, x = \beta_2, \dots, x = \beta_r$ . On the other portions of the system of curves there are to be only a finite number of vertical tangents. Hence by any line  $x = x_0, x_0 \neq \beta_1, \dots, \beta_r$ , the system of curves will be cut in only a finite number of points.

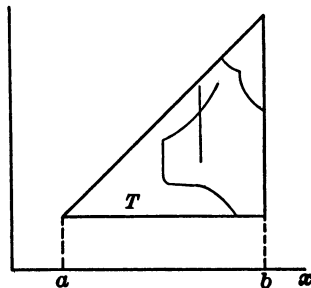


FIG. 2.

*Condition (B).* In the region  $t: a \leq x \leq b$ , a real function of the single variable  $x$  is to be continuous except at a finite number of points,  $\gamma_1, \dots, \gamma_r$ , and is to remain finite.

### § 1. A General Theorem.

The existence of the unique continuous solution of (1) when  $\phi(x)$  and the kernel are continuous is established by the development of  $u(x)$  into a series by substitution of the equation into itself.\* If we assume for the moment that the equation (1) has a continuous solution,  $u(x)$ , then the following equations will hold:

$$u(x) = \phi(x) + \int_a^x K(x, \xi) u(\xi) d\xi,$$

$$u(x) = \phi(x) + \int_a^x K(x, \xi) \phi(\xi) d\xi + \int_a^x K(x, \xi) \int_a^\xi K(\xi, \xi') u(\xi') d\xi d\xi,$$

\* Method of Successive Substitutions, LIOUVILLE: Journal de Mathématiques, vol. 2 (1837), pp. 16-36 (applied to a particular equation).

etc. As  $n$  becomes infinite, the general expression becomes an infinite series. This series is shown to be uniformly convergent, and a solution of the given integral equation (1). The same method applies when  $K(x, \xi)$  is not so much restricted.\*

Let us define besides the linear region  $t$  and the two dimensional region  $T$ , a

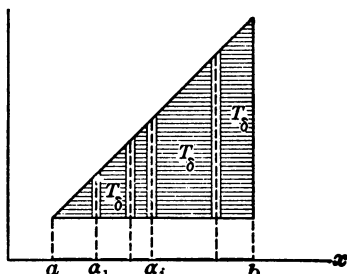


FIG. 3.

linear region  $t_\delta$  formed from  $t$  by removing the small portions  $\alpha_i - \delta < x < \alpha_i + \delta$  ( $i = 1, 2, \dots, l$ ); and a two dimensional region  $T_\delta$  formed from  $T$  by removing the small strips  $\alpha_i - \delta < x < \alpha_i + \delta$  ( $i = 1, 2, \dots, l$ ), where the  $\delta$  is an arbitrarily small magnitude, and the  $\alpha$ 's are finite in number.

**THEOREM.** Consider the integral equation (1). Let  $K(x, \xi)$  satisfy Condition (A), and  $\phi(x)$  Condition (B). Furthermore let us assume

the following conditions:

(a)  $\int_a^x |K(x, \xi)| d\xi$  converges for all values of  $x$  in the interval  $a \leq x \leq b$  except for a finite number of such values,†  $x = \lambda_1, \lambda_2, \dots, \lambda_l$ , and the function thus defined remains finite, less than  $M$ ;

(b) Let  $\epsilon$  and  $\delta$  be two arbitrary positive quantities. Then there is a positive quantity  $\eta$  and a region  $T_\delta$  such that

$$\int_x^{x+\eta} |K(x, \xi)| d\xi < \epsilon, \quad (x, y) \text{ and } (x, y + \eta) \text{ in } T_\delta.$$

The points  $\alpha_1, \dots, \alpha_l$  that appear in the definition of  $T_\delta$  are independent of  $\epsilon$  and  $\delta$  and include the points  $\beta$  of (A) and  $\lambda$  of (a);

(c)  $t$  can be divided into  $k$  parts, bounded by the points  $a = a_0, a_1, \dots, b = a_k$ , such that

$$\int_{a_i}^x |K(x, \xi)| d\xi \leq H < 1 \quad a_i \leq x \leq a_{i+1}; \quad x \neq a_1, \dots, a_l.$$

Under these conditions there is one and only one solution  $u(x)$  of the integral equation (1), continuous in  $t$  except for a finite number of points, and these points will be among the points  $\gamma_1, \dots, \gamma_l$ , of (B), and  $\alpha_1, \dots, \alpha_l$ , of (b).

In the proof of this theorem (a) and (c) are used in showing the convergence of the expansion of the solution, and (b) in developing what continuity exists.

\*The principal theorem of this part is a generalization of theorems submitted by Dr. W. A. HURWITZ, Dr. C. N. MOORE and myself as answers to a problem given out by Professor MAXIME BÜCHER in his course on integral equations, Harvard University, 1907-08.

†An example of a point  $\lambda$  not a point  $\beta$  is given by the point  $x = 1$ , where  $K(x, \xi) = 1$  in the horizontally shaded region of the adjoined figure, and  $K(x, \xi) = 1/(x-1)$  in the vertically shaded region.

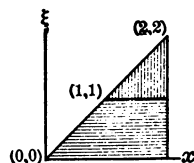


FIG. 4.

§ 2. *Convergence of the Expansion.*

We shall show that the series

$$\bar{u}(x) = \int_a^x K(x, \xi) \phi(\xi) d\xi + \int_a^x K(x, \xi) \int_a^\xi K(\xi, \xi') \phi(\xi') d\xi' d\xi + \dots$$

converges absolutely and uniformly in the region  $t_\delta$  and represents in  $t$  a function finite and continuous except for a finite number of points.

Every term of this series is continuous in the region  $t_\delta$ . For if  $r(x)$  is any function continuous in  $t$  except for a finite number of points and finite, in absolute value less than  $N$ , the integral

$$F(x) = \int_a^x K(x, \xi) r(\xi) d\xi$$

is continuous in the region  $t_\delta$  and finite in  $t$ .

To prove this take  $x_0$ , any point in  $t_\delta$ , and  $x_0 + \Delta x$ , a point near it and in  $t_\delta$ , Fig. 5, and consider

$$\begin{aligned} & \left| \int_a^{x_0 + \Delta x} K(x_0 + \Delta x, \xi) r(\xi) d\xi - \int_a^{x_0} K(x_0, \xi) r(\xi) d\xi \right| = |\Delta F| \\ & \leq \left| \int_{x_0}^{x_0 + \Delta x} K(x_0 + \Delta x, \xi) r(\xi) d\xi \right| + \left| \int_a^{x_0} \{K(x_0 + \Delta x, \xi) - K(x_0, \xi)\} r(\xi) d\xi \right| \\ & \leq N \int_{x_0}^{x_0 + \Delta x} |K(x_0 + \Delta x, \xi)| |d\xi| + N \int_a^{x_0} |K(x_0 + \Delta x, \xi) - K(x_0, \xi)| |d\xi|. \end{aligned}$$

Let us assign the value  $\rho$ , arbitrarily small, in advance. On each side of every one of the  $p$  points  $\xi_1, \xi_2, \dots, \xi_p$  in which the line  $x = x_0$  is cut by the system of curves of discontinuity of  $K(x, \xi)$  lay off a distance  $\frac{1}{2}\eta$  such that

$$\int_{\xi_i - \frac{1}{2}\eta}^{\xi_i + \frac{1}{2}\eta} |K(x, \xi)| |d\xi| < \epsilon$$

in accordance with the condition (b) of the theorem. Take

$$\epsilon = \left( \frac{\rho}{p+1} \right) \left( \frac{1}{2N} \right).$$

Take  $\Delta x < \eta$ , and small enough so that none of the curves of the system are cut by any of the lines  $\xi = \xi_i \pm \eta/2$  in the interval  $x_0 \leq x \leq x_0 + \Delta x$ . We shall then have

$$|\Delta F| \leq \frac{\rho}{2} + N \left[ \sum \int_{A_i}^{B_i} |K(x_0 + \Delta x, \xi) - K(x_0, \xi)| |d\xi| \right],$$

where the integration is now extended over the sections of the line  $x = x_0$

remaining after the  $p$  pieces have been removed. And since  $K(x, \xi)$  is continuous in these closed regions (the shaded regions in the diagram, Fig. 5),  $\Delta x$  may be chosen small enough so that

$$|K(x_0 + \Delta x, \xi) - K(x, \xi)| \leq \frac{\rho}{(x_0 - a)} \frac{1}{2N}.$$

Accordingly we have, finally,

$$|\Delta F| \leq \frac{\rho}{2} + \frac{\rho}{2(x-a)} \Sigma \{B_i - A_i\} \leq \frac{\rho}{2} + \frac{\rho}{2(x-a)} (x-a) \leq \rho;$$

and  $F$  is continuous in  $t_\delta$ , i. e. continuous in  $t$  except for the points  $\alpha_1, \dots, \alpha_l$ , since  $\delta$  is as small as we please to take it. Moreover,  $F$  is finite in  $t$  since  $|F| \leq NM$ .

The same proof applies to the function

$$R(x) = \int_a^x K(x, \xi) r(\xi) d\xi, \quad a \leq R \leq x,$$

showing that it is continuous except for the points  $\alpha$  and finite, in  $t$ .

The function

$$\int_a^x K(x, \xi) r(\xi) d\xi$$

itself satisfies the conditions laid on  $r(x)$ . For it is continuous except for the points  $\alpha$  and finite. Consequently, as is seen by mathematical induction, every term, and every integral appearing in every term, of the series is an  $r(x)$ ,

with points of discontinuity included in the points  $\alpha_1, \dots, \alpha_l$ .

We now show the absolute and uniform convergence of the series

$$(3) \quad \bar{u}(x) = \int_a^x K(x, \xi) \phi(\xi) d\xi + \int_a^x K(x, \xi) \int_a^\xi K(\xi, \xi') \phi(\xi') d\xi' d\xi + \dots$$

in the region  $t_\delta$ . If we write it as

$$\sum_{n=0}^{\infty} u_{n+1}(x)$$

its general term is

$$u_{n+1}(x) = \int_a^x K(x, \xi) \int_a^\xi K(\xi, \xi') \int_a^{\xi'} \dots \int_a^{\xi^{(n-1)'}} K(\xi^{(n-1)}, \xi^{(n)}) \phi(\xi^{(n)}) d\xi^{(n)} \dots d\xi' d\xi.$$

We have immediately

$$(4) \quad |u_{n+1}(x)| \leq N \int_a^x |K(x, \xi)| \int_a^\xi |K(\xi, \xi')| \int_a^{\xi'} \dots \int_a^{\xi^{(n-1)'}} |K(\xi^{(n-1)}, \xi^{(n)})| d\xi^{(n)} \dots d\xi' d\xi,$$

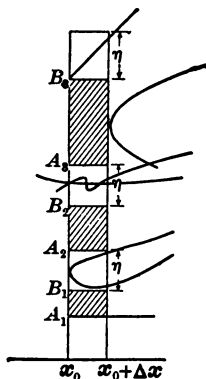


FIG. 5.

$$|u_{n+1}(x)| \leqslant$$



§ 3. *Existence of a Solution.*

If we define

$$(8) \quad U(x) = \phi(x) + \bar{u}(x)$$

we know that  $U(x)$  is a function continuous in  $t$  except for a finite number of points, which are among the points  $\gamma_1, \dots, \gamma_n, \alpha_1, \dots, \alpha_l$ ; and finite  $\leq \Sigma + N$ . It is now easy to show that  $U(x)$  is a solution of the integral equation (1).

If we write for this purpose

$$U(x) = S_n(x) + R_n(x),$$

where  $S_n(x)$  is the sum of the first  $n$  terms, the three functions  $U(x)$ ,  $S_n(x)$ ,  $R_n(x)$  are of the type  $r(x)$ , § 2, and

$$\begin{aligned} \int_a^x K(x, \xi) U(\xi) d\xi &= \int_a^x K(x, \xi) S_n(\xi) d\xi + \int_a^x K(x, \xi) R_n(\xi) d\xi \\ \text{or} \quad \int_a^x K(x, \xi) U(\xi) d\xi - \int_a^x K(x, \xi) S_n(\xi) d\xi &= \int_a^x K(x, \xi) R_n(\xi) d\xi. \end{aligned}$$

Hence

$$\lim_{n=\infty} \left( \int_a^x K(x, \xi) U(\xi) d\xi - \int_a^x K(x, \xi) S_n(\xi) d\xi \right) = 0$$

for

$$\left| \int_a^x K(x, \xi) R_n(\xi) d\xi \right| \leq \int_a^x |K(x, \xi)| |R_n(\xi)| d\xi;$$

and since  $|R_n(x)|$  can be made less than an arbitrarily assigned  $\epsilon$  by a suitable choice of  $n$ , we have

$$\left| \int_a^x K(x, \xi) R_n(\xi) d\xi \right| \leq \epsilon \int_a^x |K(x, \xi)| d\xi,$$

which in  $t_0$  is  $\leq \epsilon M$  and can therefore be made as small as we please by taking  $n$  large enough.

Hence

$$\int_a^x K(x, \xi) U(\xi) d\xi = \lim_{n=\infty} \int_a^x K(x, \xi) S_n(\xi) d\xi$$

and we can integrate the series term by term. We see in this way that  $U(x)$  and

$$\phi(x) + \int_a^x K(x, \xi) U(\xi) d\xi$$

are identical, and that consequently  $U(x)$  is a solution of the integral equation (1).

#### § 4. Uniqueness of the Solution.

If  $u(x)$  is any solution whatever, finite, and continuous in  $t$  except for a finite number of points,

$$v(x) = u(x) - U(x)$$

will be a function of the same kind, and since

$$u(x) = \phi(x) + \int_a^x K(x, \xi) u(\xi) d\xi$$

and

$$U(x) = \phi(x) + \int_a^x K(x, \xi) U(\xi) d\xi$$

$v(x)$  will be a solution of the homogeneous integral equation

$$v(x) = \int_a^x K(x, \xi) v(\xi) d\xi.$$

Consequently

$$v(x) = \int_a^x K(x, \xi) \int_a^\xi K(\xi, \xi') \int_a^{\xi'} \dots \int_a^{\xi^{(n-1)}} K(\xi^{(n-1)}, \xi^{(n)}) d\xi^{(n)} \dots d\xi, \\ n = 1, 2, \dots,$$

and

$$|v(x)| \leq N \int_a^x |K(x, \xi)| \int_a^\xi |K(\xi, \xi')| \int_a^{\xi'} \dots \int_a^{\xi^{(n-1)}} |K(\xi^{(n-1)}, \xi^{(n)})| d\xi^{(n)} \dots d\xi.$$

But by the analysis of § 2, this last inequality leads to

$$|v(x)| \leq N \{ (n+1) M \}^{k-1} H^{n+2-k}, \quad x \text{ in } t_\delta,$$

and since  $H < 1$  and  $n$  is arbitrary, the right hand side can be made as small as we please. Hence for any point of  $t_\delta$

$$v(x) = 0$$

and  $U(x)$  is the only solution of the given type. Thus the theorem of § 1 is proved.

A few examples will show its scope. A kernel  $K(x, \xi)$  that satisfies the given conditions is  $1/2(x-a)$ . Here

$$\int_a^x |K(x, \xi)| d\xi = \frac{1}{2},$$

and we know that there is one and only one finite solution of

$$u(x) = \phi(x) + \int_a^x \frac{u(\xi)}{2(x-a)} d\xi$$

through any finite range,  $a$  to  $b$ , whatever. A more complicated example of the same kind is

$$|K(x, \xi) = \left\{ \begin{array}{ll} \frac{1}{5(x-a)}, & \cdot \cdot \cdot \cdot \cdot \cdot, \quad a \leq x \leq x_0; \\ \frac{1}{5(x_0-a)}, & \cdot \cdot \cdot \cdot, \quad a \leq \xi \leq x_0 \\ \frac{1}{5(x_0-a)} + \frac{1}{2(x-x_0)}, & x_0 < \xi \leq x \end{array} \right\} x_0 < x \leq b.$$

Other examples, of theoretical importance, will appear later. An example where the theorem does not hold we get by putting  $K(x, \xi) = 2/(x-a)$ . This fails to satisfy (c), for

$$\int_a^x |K(x, \xi)| d\xi = [2$$

and we can choose no first interval small enough so that

$$\int_a^x |K(x, \xi)| d\xi \leq H < 1$$

within it. As a matter of fact, in this case there is more than one finite solution. A second case, where (c) is satisfied but where (a) fails to be, is obtained by choosing

$$K(x, \xi) = \left\{ \begin{array}{ll} \frac{1}{5(x-a)}, & a \leq x \leq x_0; \\ \frac{1}{5(\xi-a)}, & x_0 < x \leq b. \end{array} \right.$$

### § 5. Related Theorems.

There are other theorems more or less closely related to the one that we have developed, as extensions or as special cases. Practically the same proof holds when we replace (A), (B), (a), (b), (c), by:

(A')  $K(x, \xi)$  is continuous in general\* in  $T$  except in the neighborhood of a system of curves of the kind specified in Condition (A).

(B')  $\phi(x)$  is continuous in general in  $t$  except in the neighborhood of a finite number of points, and is finite.

\* By " $K(x, \xi)$  is continuous in general except in the neighborhood of a system of curves of the kind specified in Condition (A)," we mean that there exists a finite number of curves of the kind specified in Condition (A) such that if every one is enclosed by a strip of width  $2\rho$ , i. e.,  $\rho$  on each side, where  $\rho$  is arbitrarily small, and a region  $T_\rho$  formed from  $T$  by taking away these strips, then  $K(x, \xi)$  will be continuous throughout  $T_\rho$  except for a finite number of curves of the kind specified in Condition (A). An analogous meaning is attached to " $\phi(x)$  continuous in general in  $t$  except in the neighborhood of a finite number of points." Thus  $1/\sin 1/x$  is "continuous in general except in the neighborhood" of the point  $x = 0$ , Fig. 6.

(a') The integral

$$\int_a^x |K(x, \xi)| d\xi$$

converges in  $t$  except in the neighborhood of a finite number of points  $\lambda_1, \dots, \lambda_l$ , and remains finite.

(b') There is a finite number of points  $\alpha_1, \dots, \alpha_l$  such that when the region  $t_\delta$  is constructed there remain in it merely a finite number of points  $\alpha'_1, \dots, \alpha'_l$ ,  $l'$  not necessarily independent of  $\delta$ , such that when  $\epsilon$  and  $\delta'$ , ( $0 < \delta' \leq \delta$ ), are chosen at pleasure and  $t_{\delta'}$  constructed by removing from  $t_\delta$  regions of width  $2\delta'$  containing respectively the points  $\alpha_i$ , there is a length  $\eta$  for which

$$\int_y^{y+\eta} |K(x, \xi)| d\xi < \epsilon \quad \begin{cases} x \text{ in } t_{\delta'} \\ (x, y) \text{ and } (x, y + \eta) \text{ in } T. \end{cases}$$

(c')  $t$  can be divided into  $k$  parts, bounded by points  $a = a_0, a_1, \dots, a_{k-1}, b = a_k$  such that

$$\int_{a_i}^x |K(x, \xi)| d\xi \leq H < 1 \quad \begin{cases} a_i \leq x \leq a_{i+1} \\ H \text{ independent of } \delta, \delta'. \end{cases}$$

The conclusion in this case is that there is one and only one solution of the equation continuous in  $t$  except in the neighborhood of a finite number of points.

An obvious special case of the theorem of § 1 is where the kernel is continuous throughout  $T$ . Another is the equation

$$u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{(x - \xi)^\lambda (x - a)^\mu (\xi - a)^\nu} u(\xi) d\xi,$$

where  $K(x, \xi)$  is continuous throughout  $T$ , and  $\lambda + \mu + \nu < 1$ .

The condition (A) can be replaced by conditions on the integral of the kernel; for instance (A) and (b) can together be replaced by the condition which follows:

The integral

$$\int_a^x K(x, \xi) r(\xi) d\xi$$

where  $r(x)$  is finite in  $t$  and continuous except for a finite number of points, shall converge except at most for a finite number of values of  $x$ , and the function of  $x$  thus defined shall remain finite; furthermore, it shall be continuous except at most for a finite number of values of  $x$ , denoted by  $\alpha_1, \dots, \alpha_l$ , which are independent of the choice of  $r(x)$ .

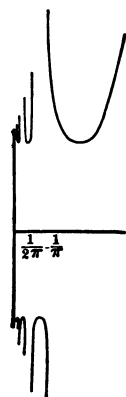


FIG. 6.

A special case of this theorem, elegant on account of the simplicity of the

conditions imposed, has been treated by Dr. W. A. HURWITZ.\* The hypotheses for this case were:

(B'')  $\phi(x)$  continuous in  $t$ ;

(a'')  $\int_a^x |K(x, \xi)| d\xi$  converges in  $t$ ;

(b'')  $\int_a^x |K(x, \xi)| d\xi$  represents a continuous function in  $t$ ;

(c'')  $|K(x_1, \xi)| \cong |K(x_2, \xi)|$  when  $x_1 > x_2$ .

Here (a'') implies (a) of § 1, and (b'') and (c'') imply (b) and (c). For since

$$|K(x_1, \xi)| \cong |K(x_2, \xi)|, \quad x_1 > x_2,$$

$$\int_{a_1}^x |K(x, \xi)| d\xi \leq \int_{a_1}^x |K(b, \xi)| d\xi$$

and

$$\int_y^{y+\eta} |K(x, \xi)| d\xi \leq \int_y^{y+\eta} |K(b, \xi)| d\xi,$$

and it follows from the convergence of

$$\int_a^x |K(b, \xi)| d\xi$$

that both (b) and (c) must hold.

### § 6. Application to a Non-Integrable Kernel.

An interesting application of the general theorem is to integral equations where the kernels have such singularities that they are no longer integrable. Let us consider the equation

$$(9) \quad u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)} u(\xi) d\xi,$$

where  $K(x, \xi)$  satisfies Condition (A) and is finite, where  $\phi(x)$  satisfies Condition (B) and vanishes at  $x = a$  in a way to be explained below, and where  $f(\xi)$  is continuous in  $t$  and unequal to zero except at the point  $a$ , where it may vanish in any way. We assume for convenience that

$$\lim_{y \rightarrow a} \int_y^x \frac{d\xi}{f(\xi)}$$

---

\*As a problem in Professor BÔCHER's course in integral equations: Harvard University, 1907-08. Dr. HURWITZ also suggests the theorem: If

(1)  $K(x, \xi)$  increases with  $x$  for any value of  $\xi$ ;

(2)  $\int_a^x |K(x, \xi)| d\xi$  converges and represents a continuous function

(3)  $\phi(x)$  is finite and integrable;

then the equation has one and only one finite and integrable solution.

does not exist; otherwise the case comes under the theorem of § 1. Without loss of generality we may take

$$|K(x, \xi)| \leq H < 1,$$

putting whatever multiplicative constant is necessary into the denominator with  $f(\xi)$ .

If we transform this equation by means of the substitution

$$(10) \quad u(x) = r(x)v(x)$$

we obtain

$$r(x)v(x) = \phi(x) + \int_a^x \frac{K(x, \xi)r(\xi)}{f(\xi)} v(\xi) d\xi,$$

so that

$$v(x) = \frac{u(x)}{r(x)}$$

will be a solution of the equation

$$(11) \quad v(x) = \frac{\phi(x)}{r(x)} + \int_a^x K(x, \xi) \frac{r(\xi)}{f(\xi)r(x)} v(\xi) d\xi;$$

and conversely, if  $v(x)$  is a solution of this last equation,

$$u(x) = r(x)v(x)$$

will be the solution of the first.

Now let us see if we can so choose  $r(x)$  that the kernel in equation (11) shall satisfy the conditions (a), (b), (c) of the fundamental theorem. That is,  $r(\xi)$  will have to be such a function that it will not only erase the singularity due to  $f(\xi)$ , but will also have enough vanishing left so that by means of the integration it will suffice also for the singularities due to  $r(x)$ . Obviously, no function with a vanishing point of finite order at  $a$ , except in the single case where  $f(\xi)$  vanishes to the first order (a case that is taken care of also by the succeeding formulas), will suffice, for the order of vanishing of its integral is greater only by one. And therefore we must look for functions that have essential singularities at the point  $a$ . A satisfactory choice is

$$(12) \quad r(x) = e^{-\int_a^x \frac{dx}{|f(x)|}}.$$

With the substitution defined by this value of  $r(x)$ , the equation for  $v(x)$  is

$$(13) \quad v(x) = \phi(x) e^{\int_a^x \frac{dx}{|f(x)|}} + \int_a^x K(x, \xi) \frac{e^{\int_a^x \frac{dx}{|f(x)|}} e^{-\int_a^\xi \frac{d\xi}{|f(\xi)|}}}{f(\xi)} d\xi,$$

and the new kernel is

$$(14) \quad \bar{K}(x, \xi) = K(x, \xi) \frac{e^{\int_x^b \frac{dx}{|f(x)|} - \int_\xi^b \frac{d\xi}{|f(\xi)|}}}{f(\xi)}.$$

This new kernel  $\bar{K}$  satisfies the conditions (a), (b), (c). For the integral

$$\int_a^x |\bar{K}(x, \xi)| d\xi$$

converges,  $a \leq x \leq b$ , except for a finite number of points  $\lambda_1, \dots, \lambda_i$  and is  $< M$ ,  $x \neq \lambda_1, \dots, \lambda_i$ .

Since  $|K(x, \xi)| \leq H$  we see, when we have defined the points  $\alpha_1 \dots \alpha_i$ , that there is a length  $\eta$  such that

$$\int_{\eta}^{\eta+\eta} |\bar{K}(x, \xi)| d\xi < \epsilon, \quad (x, y) \text{ and } (x, y + \eta) \text{ in } T_s.$$

And finally we can show that

$$\int_a^x |\bar{K}(x, \xi)| d\xi \leq H < 1, \quad \left\{ \begin{array}{l} a \leq x \leq b, \\ x \neq \alpha_1, \dots, \alpha_i. \end{array} \right.$$

For

$$\int_a^x |K(x, \xi)| \left| \frac{e^{\int_x^b \frac{d\xi}{|f(x)|} - \int_\xi^b \frac{d\xi}{|f(\xi)|}}}{f(\xi)} \right| d\xi \leq H \int_a^x \frac{e^{\int_x^b \frac{dx}{|f(x)|} - \int_\xi^b \frac{d\xi}{|f(\xi)|}}}{|f(\xi)|} d\xi.$$

In this integral let

$$\zeta = \int_\xi^b \frac{d\xi}{|f(\xi)|} \quad \text{and} \quad z = \int_x^b \frac{dx}{|f(x)|};$$

then

$$d\zeta = - \frac{d\xi}{|f(\xi)|}.$$

When  $\xi = a$ , we have  $\zeta = +\infty$ , and when  $\xi = x$ , we have  $\zeta = z$ , so that

$$\int_a^x \frac{e^{\int_x^b \frac{dx}{|f(x)|} - \int_\xi^b \frac{d\xi}{|f(\xi)|}}}{|f(\xi)|} d\xi$$

becomes

$$\int_z^\infty e^{z-\zeta} d\zeta = [-e^{z-\zeta}]_z,$$

which has the value 1. Whence

$$\int_a^x |\bar{K}(x, \xi)| d\xi \leq H.$$

Hence there is one and only one  $v(x)$  which is a solution of (13) finite and continuous in  $t$  except for a finite number of points, provided that

$$\phi(x)e^{\int_x^b \frac{dx}{|f(x)|}}$$

remains finite as  $x$  approaches  $a$ . Hence by (10) there is at least one solution of the given equation (9), provided that

$$\phi(x)e^{\int_x^b \frac{dx}{|f(x)|}}$$

remains finite, namely,

$$(15) \quad u(x) = e^{-\int_x^b \frac{dx}{|f(x)|}} v(x).$$

It vanishes at the point  $a$  at least as sharply as

$$\text{const. } e^{-\int_x^b \frac{dx}{|f(x)|}}.$$

The case where  $f(\xi)$  is discontinuous at a finite number of points is no more general than this that we have treated, for the discontinuities may be taken up by the  $K(x, \xi)$ . The case where  $f(\xi)$  vanishes again in the interval is more complex. If it vanishes as a result of a finite jump, the problem may be treated by the previous analysis if we replace the equation (13) by

$$v(x) = \bar{\phi}(x) + \int_{a_1}^x K(x, \xi) \frac{e^{\int_x^b \frac{dx}{|f(x)|} - \int_{\xi}^b \frac{d\xi}{|f(\xi)|}}}{f(\xi)} v(\xi) d\xi,$$

where  $a_1$  is this second vanishing point of  $f(\xi)$  and where

$$\bar{\phi}(x) = \phi(x)e^{\int_x^{a_1} \frac{dx}{|f(x)|}} + \int_a^{a_1} K(x, \xi) \frac{e^{\int_x^{a_1} \frac{dx}{|f(x)|} - \int_{\xi}^{a_1} \frac{d\xi}{|f(\xi)|}}}{f(\xi)} v(\xi) d\xi,$$

$v(\xi)$  being known in the interval  $(a, a_1)$  as a solution of the equation just studied.

### § 7. A Generalization of Equation (9).

An equation closely similar to (9) is

$$(16) \quad u(x) = \phi(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)g(x)} u(\xi) d\xi,$$

where the continuous functions  $f(x)$ ,  $g(x)$ , which vanish in  $t$  only at  $x = a$ , are such that the limit

$$\lim_{y \rightarrow a} \int_y^x \frac{d\xi}{f(\xi)g(\xi)}$$

does not exist. If we multiply through by  $g(x)$  we get the equation

$$u(x)g(x) = \phi(x)g(x) + \int_a^x \frac{K(x, \xi)}{f(\xi)} u(\xi) d\xi,$$

and if we put  $w(x) = u(x)g(x)$ ,

$$w(x) = \phi(x)g(x) + \int_a^x \frac{K(x, \xi)}{g(\xi)f(\xi)} w(\xi) d\xi.$$

This equation has a finite solution  $w(x)$  continuous in  $t$  except for a finite number of points, if

$$\phi(x)g(x)e^{\int_x^b \frac{dx}{|f(x)| |g(x)|}}$$

remains finite as  $x$  approaches  $a$ . Moreover, at the point  $a$ ,  $w(x)$  vanishes as strongly as

$$\text{const.} \cdot e^{-\int_x^b \frac{dx}{|f(x)| |g(x)|}}.$$

Hence the  $u(x)$ , which is given by

$$u(x) = \frac{w(x)}{g(x)},$$

vanishes at the point  $a$  as strongly as

$$\text{const.} \cdot \frac{1}{g(x)} e^{-\int_x^b \frac{dx}{|f(x)| |g(x)|}}.$$

### § 8. A Generalization of Equation (16).

Finally let us investigate the equation

$$(17) \quad u(x) = \phi(x) + \int_a^x \frac{K(x, \xi) u(\xi)}{g(x)f(\xi) \prod_{i=1}^p [\xi - \psi_i(x)]^{\lambda_i}} d\xi,$$

where

$$\sum_{i=1}^p \lambda_i = \lambda < 1$$

and the  $\psi_i$  are various arbitrary continuous functions.

This equation is equivalent to the equation

$$(18) \quad w(x) = \theta(x) + \int_a^x \frac{\alpha K(x, \xi)}{h(\xi) \prod_{i=1}^p [\xi - \psi_i(x)]^{\lambda_i}} w(\xi) d\xi,$$

for (17) is reducible to (18) by the substitutions

$$w(x) = u(x)g(x), \quad \theta(x) = \phi(x)g(x), \quad h(x) = \alpha f(x)g(x);$$

and conversely, (18) is reducible to (17) by the inverse substitutions. We shall suppose, as we can without loss of generality, that  $|K(x, \xi)| \leq H < 1$ . The  $\alpha$  we shall leave undetermined for the present.

By the substitution  $w(x) = r(x)v(x)$  the equation (18) is equivalent to

$$(19) \quad v(x) = \frac{\theta(x)}{r(x)} + \int_a^x \frac{r(\xi)}{h(\xi)r(x)} \frac{\alpha K(x, \xi)}{\prod_{i=1}^p [\xi - \psi_i(x)]^{\lambda_i}} v(\xi) d\xi.$$

Let us denote by  $\nu$  any positive number we please less than  $1 - \lambda$ , and choose  $r(x)$  so that

$$\int_a^x \left| \left( \frac{r(\xi)}{h(\xi)r(x)} \right)^{\frac{\nu+\lambda}{\nu}} \right| d\xi$$

shall be convergent. Such a choice is

$$r(x) = e^{-\int_x^b \frac{ds}{|h(s)|^{(\nu+\lambda)/\nu}}}.$$

For if we make the substitutions

$$z = \int_x^b \frac{dx}{|h(x)|^{(\nu+\lambda)/\nu}}, \quad \zeta = \int_\xi^b \frac{d\xi}{|h(\xi)|^{(\nu+\lambda)/\nu}}, \quad d\zeta = -\frac{d\xi}{|h(\xi)|^{(\nu+\lambda)/\nu}},$$

the given integral reduces to

$$-(e^s)^{\frac{\nu+\lambda}{\nu}} \int_{\lim_{z=a}}^s (e^{-\zeta})^{\frac{\nu+\lambda}{\nu}} d\zeta = (e^s)^{\frac{\nu+\lambda}{\nu}} \frac{\nu}{\nu+\lambda} \int_{\frac{\nu+\lambda}{\nu}s}^{\lim_{z=a} s} e^{-\eta} d\eta = 1 - e^{\frac{\nu+\lambda}{\nu}s} \left( e^{-\frac{\nu+\lambda}{\nu} \lim_{z=a} s} \right).$$

But  $z \leq \lim_{x=a} z$  when  $x \geq a$ , so that the second term of this last expression for the integral is less than or equal to 1, and for the given choice of  $r(\xi)$  we have

$$\int_a^x \left| \frac{r(\xi)}{h(\xi)r(x)} \right|^{\frac{\nu+\lambda}{\nu}} d\xi < 1.$$

It must be remembered that  $r(x)$ , by its definition, depends on  $\alpha$ .

Let us now consider the integral of the absolute value of the kernel of equation (19) and show that with this definition of  $r(x)$  it can be made  $< 1$  throughout  $t$  by properly choosing the constant  $\alpha$ .

It is obvious that if  $\alpha, b, m, n$  are any numbers whatever not negative,

$$\alpha^m b^n \leq \alpha^{m+n} + b^{m+n};$$

and that if  $\nu$  is any positive number

$$\alpha^m b^n \leq \alpha^{m+n/\nu} + b^{m\nu+n}.$$

Moreover if  $\alpha, m, b_1, \dots, b_p, n_1, \dots, n_p$ , are any numbers whatever not negative,

$$\alpha^m b_1^{n_1} b_2^{n_2} \dots b_p^{n_p} \leq \alpha^{m + \frac{n_1 + \dots + n_p}{\nu}} + b_1^{m\nu+n_1+\dots+n_p} + b_2^{m\nu+n_1+\dots+n_p} + \dots + b_p^{m\nu+n_1+\dots+n_p}.$$

Now

$$\begin{aligned} \int_a^x \left| \left( \frac{r(\xi)}{h(\xi)r(x)} \right)^{\frac{\nu+\lambda}{\nu}} \frac{\alpha K(x, \xi)}{\prod_{i=1}^p \{\xi - \psi_i(x)\}^{\lambda_i}} \right| d\xi &\leq \alpha H \int_a^x \left| \frac{r(\xi)}{h(\xi)r(x)} \right| \prod_{i=1}^p \left| \frac{1}{[\xi - \psi_i(x)]^{\lambda_i}} \right| d\xi \\ &\leq \alpha H \int_a^x \left| \frac{r(\xi)}{h(\xi)r(x)} \right| \prod_{i=1}^p \left| \frac{1}{\xi - \psi_i(x)} \right|^{\lambda_i} d\xi. \end{aligned}$$

The integrand now, however, is a product of which the factors are all positive. We have then, according to what we have just seen (putting  $m = 1$ ),

$$\left| \frac{r(\xi)}{h(\xi)r(x)} \right| \prod_{i=1}^p \left| \frac{1}{\xi - \psi_i(x)} \right|^{\lambda_i} \leq \left| \frac{r(\xi)}{h(\xi)r(x)} \right|^{1+\lambda/\nu} + \sum_{i=1}^p \left| \frac{1}{\xi - \psi_i(x)} \right|^{\nu+\lambda},$$

and the given integral is less than or equal to

$$\alpha H \int_a^x \left| \frac{r(\xi)}{h(\xi)r(x)} \right|^{\frac{\nu+\lambda}{\nu}} d\xi + \sum_{i=1}^p \int_a^x \left| \frac{1}{\xi - \psi_i(x)} \right|^{\nu+\lambda} d\xi.$$

Since  $\nu + \lambda < 1$  we can choose

$$P \geq \sum_{i=1}^p \int_a^x \left| \frac{1}{\xi - \psi_i(x)} \right|^{\nu+\lambda} d\xi, \quad x \text{ in } t,$$

so that the integral of the absolute value of the kernel is

$$\leq \alpha H(1 + P), \quad x \text{ in } t,$$

the  $\alpha$  being involved only explicitly; and by choosing

$$\alpha = \frac{1}{1 + P}$$

this becomes  $\leq H < 1$ . There is then under these conditions a solution of equation (19) if  $\theta(x)/r(x)$  remains finite.

In regard to (17) we know that there is one solution continuous except at a finite number of points provided that

$$\frac{\phi(x)g(x)}{r(x)}$$

remains finite as  $x$  approaches  $a$ .

The number of solutions of such equations as these — (9), (16), (17) — we have not investigated. Conceivably there may be, often in fact there are, other finite solutions.

HARVARD UNIVERSITY,  
February, 1910.